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Aerodynamic Characteristics of a Slender Body Traveling in a Tube

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Incompressible inviscid slender-body theory is applied to determine the flow about a slender body of revolution traveling in a tube. Formulas for all the static and dynamic stability derivatives are derived for an arbitrary body of revolution in terms of its cross sectional area distribution. These formulas are specialized to an ellipsoid of revolution as an illustrative example and plots of the results are presented as a function of the ratio of the maximum cross-sectional area of the body to the area of the tube. For the body whose diameter is a large percent of the tube diameter, the stability derivatives become an order of magnitude greater than they are for the same body in free air. Furthermore, a statically unstable force of attraction to the wall caused by proximity to the wall is present which does not exist at all for the body in free air. The inherent aerodynamic instability of a body in free air without controls is thus increased by the presence of the tube walls, and the walls may be said to exert a large effect on the aerodynamic characteristics of the body.

Nomenclature

a = pivot point
C_L = L/qS_m = lift coefficient
C_M = M/qS_ml = moment coefficient
C_p = pressure coefficient
$f(x)$ = see r_0
F = source strength
l = length of body
L = lift
M = pitching moment
q = $\frac{1}{2}\rho U^2$ = dynamic pressure
r = radial coordinate
r_0 = $\epsilon f(x)$ = body radius
R = tube radius
S_m = maximum cross-sectional area of body
S_0 = πr_0^2 = cross-sectional area of body
S_R = πR^2 = cross-sectional area of tube
t = time
U = freestream velocity
$\mathbf{V}$ = velocity vector
x = longitudinal coordinate
z_0 = heave displacement
α = S_0/S_R
α_m = S_m/S_R
β = potential due to heave displacement

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γ = potential due to heave rate
δ = angle of incidence
ϵ = body thickness ratio
θ = azimuthal coordinate
λ = potential due to pitch rate
μ = Fourier frequency
ξ = dummy x variable
ρ = fluid density
ϕ = potential due to thickness
ψ = potential due to pitch angle

Subscripts and superscripts

$(\ )_x$ = partial derivative with respect to x
$(\ )_r$ = partial derivative with respect to r
$(\ )_\theta$ = partial derivative with respect to θ
$(\ )_t$ = partial derivative with respect to t
$(\dot{\ })$ = derivative with respect to t
$(\ )'$ = derivative with respect to x

Introduction

UNDER a program sponsored by the Department of Transportation, aerodynamic stability problems associated with a tube transportation system are being studied. The analysis of the stability and control of such a vehicle must necessarily account for all the forces which act on the vehicle. These may conveniently be divided into three types: forces caused by propulsion, forces caused by the support system, and forces caused by the flow of air around the body. The last of these may be called the aerodynamic forces. Aerodynamic interference effects between all three types of forces can be expected. Nevertheless, as a first approximation it is proper to examine the three types of forces as though they were independent of one another. This paper will be concerned only with the flow around the body and the aerodynamic forces and moments which result therefrom.

The general shape of a vehicle traveling in a tube will certainly be long and thin, i.e., it will have a small thickness ratio. The presence of the tube walls will alter the flow properties from that which would occur if the same body were traveling in free air. For purposes of keeping the analytical work tractable it will be assumed that the fluid is inviscid and incompressible and that the vehicle can be represented by a body of revolution. It will also be assumed that the tube is infinitely long (no end effects) and has a circular cross section. An indication will be given, however, as to how the analysis would proceed if either the body or the tube or both had a noncircular cross section. The longitudinal distribution of cross-sectional area of the body of revolution will remain arbitrary during the analysis.

There are three lengths involved in this problem: $2R$, the tube diameter; $2r_0$, the body diameter; and l, the body length. For a slender body the thickness ratio $2r_0/l$ must be small. It will also be assumed that the ratio $2R/l$ is small. No mathematical restriction will be placed on the ratio of the two diameters, r_0/R, except, of course, that it be smaller than unity. On the other hand this ratio is limited by the physical assumptions that the fluid is incompressible and inviscid. Since $2r_0/l$ and $2R/l$ are both small slender-body theory will be used on the body-tube combination, but the form that the theory takes for internal flows will be found to be unconventional.

It is worth pointing out that Newman[1] has recently calculated the forces on a slender body of revolution moving parallel to an infinite straight wall. He took the ratio of separation distance to body length to be small, and was, thereby, also led to use slender-body theory for the body-wall combination.

The paper by Levine[2] should also be mentioned. This author considered the incompressible inviscid axially symmetric flow about a body in a tube, but in contrast to the present theory he did not assume the body-tube combination to be slender. The strength of the singularities was specified and then the geometry of the body was determined a posteriori.

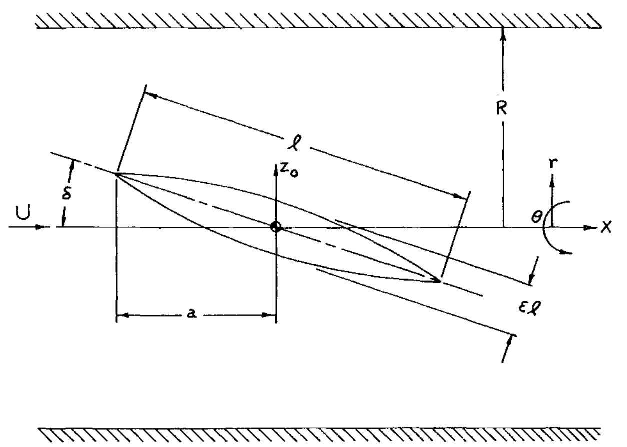

Fig. 1 Diagramatic sketch of body showing coordinate system.

Finally, one of the problems considered by Segel[3] is a circular cylinder oscillating laterally inside another cylinder. The fluid between the cylinder was taken to be viscous and a formula for the force on the inner cylinder was derived and specialized to the case of large Reynolds numbers (small viscosity). This force consists of an in-phase component which is of the added mass type and an out-of-phase component which is a damping force. Part of Segel's added mass type force per unit length is independent of Reynolds number and that part could also be obtained from the inviscid theory presented here.† On the other hand, the damping force per unit length obtained by Segel is caused completely by viscous effects, whereas the damping obtained here is inviscid and is caused by the axial variation of the body cross section. The ratio of Segel's damping force per unit length to the one derived here can be shown to be of the order $(k/Re\epsilon^2)^{1/2}$. Here k is the reduced frequency, $\omega l/U$; Re is the Reynolds number, Ul/ν; and ϵ is the thickness ratio of the body. Typical full-scale values of these quantities are $k = 1.0$, $Re = 10^8$, $\epsilon = 0.10$, and hence the viscous damping force per unit length is of the order 10^{-3} times the inviscid damping force per unit length and consequently, with air as the working fluid, viscous damping may ordinarily be safely ignored.

Formulation of the Problem

The configuration to be considered consists of a body of revolution traveling forward and at the same time slowly oscillating within the confines of a circular cylindrical tube (see Fig. 1). The problem will be formulated in "tube coordinates," i.e., in coordinates fixed with respect to the tube but moving forward with the body.

Recently a paper appeared which explicitly formulates the boundary condition and pressure on the body in coordinates aligned with the body motion (wind coordinates) for a body in free air.[4] These authors consider a body of revolution moving forward and undergoing pitching oscillations. For the body in the tube in tube coordinates this must be generalized to include heaving oscillations as well. (The difference between wind coordinates and tube coordinates is that for wind coordinates the axes are aligned with the total motion including heave, whereas for tube coordinates the axes are aligned only with the forward motion and heave must be considered as a perturbation.) The formulation presented in Ref. 4 can be used as a guide.

Consider a slowly oscillating body of revolution. The body surface may be described by the equation $S(x,r,\theta,t) = 0$.

† Added mass forces are ordinarily ignored in determining the control characteristics of a body in air.

The boundary condition of the body surface requires that the substantial derivative of S vanish there:

$$\partial S/\partial t + \mathbf{V}\cdot \text{grad } S = 0 \quad \text{at } S = 0 \tag{1}$$

For a body of revolution performing simultaneously a small amplitude pitching oscillation about a pivot at $x = a$ together with a heaving oscillation, the equation of the body surface may be approximated to the first order by

$$r = \epsilon f(x) - [\delta(t)(x - a) - z_0(t)] \cos\theta \tag{2}$$

Here ϵ is taken to be small so that $f(x)$ is of order unity; (see Fig. 1 for definitions). The effect of heave displacement is nil in free air, but in the tube it connotes proximity to the wall and, as such, produces a force.

For irrotational flow there exists a potential function $\Omega(x,r,\theta,t)$ whose gradient yields the velocity vector;

$$\mathbf{V} = \text{grad } \Omega \tag{3}$$

A perturbation velocity potential is now introduced,

$$\Omega = U[x + \Phi(x,r,\theta,t)] \tag{4}$$

Thus, the velocity components are:

$$\text{axial: } u/U = 1 + \Phi_x; \quad \text{radial: } v/U = \Phi_r \qquad \text{azimuthal: } w/U = (1/r)\Phi_\theta \tag{5}$$

For low-frequency oscillations the perturbation potential may be expressed as the sum

$$\Phi = \phi(x,r) + [\delta\psi(x,r) + \dot{\delta}\lambda(x,r) + z_0\beta(x,r) + \dot{z}_0\gamma(x,r)] \cos\theta \tag{6}$$

Then, in terms of $\phi,\psi,\lambda,\beta,\gamma$ the flow tangency condition, Eq. (1), becomes

$$\{[\dot{\delta}(x - a) - \dot{z}_0]/U\} \cos\theta + [-\epsilon f'(x) + \delta \cos\theta] \times (1 + \phi_x + \delta\psi_x \cos\theta + \dot{\delta}\lambda_x \cos\theta + z_0\beta_x \cos\theta + \dot{z}_0\gamma_x \cos\theta) + (\phi_r + \delta\psi_r \cos\theta + \dot{\delta}\lambda_r \cos\theta + z_0\beta_r \cos\theta + \dot{z}_0\gamma_r \cos\theta) = 0 \tag{7}$$

Equation (7) must, of course, be satisfied on the body, i.e., at r as given by Eq. (2). Since the location of the body in tube coordinates involves the quantities δ and z_0, it is not possible to derive explicit formulas for the boundary conditions for the various potentials unless Eq. (7) is expanded about $r = \epsilon f(x)$ for small δ and z_0. Any function F when expanded in this way becomes

$$F = F|_{r=\epsilon f} - [\delta(t)(x - a) - z_0(t)] \cos\theta \partial F/\partial r|_{r=\epsilon f} \tag{8}$$

Applying Eq. (8) to Eq. (7) leads to the proper boundary condition at $r = \epsilon f$. Upon equating coefficients of $\delta,\dot{\delta},z_0,\dot{z}_0$ it follows that the various potentials satisfy the following set of boundary conditions:

$$\begin{gathered}\phi_r = \epsilon f'(x)(1 + \phi_x) \equiv r_0'(x)(1 + \phi_x) \\ \psi_r = -(1 + \phi_x) + (x - a)\phi_{rr}, \ \lambda_r = -(x - a)/U \\ \beta_r = -\phi_{rr}, \ \gamma_r = 1/U\end{gathered} \tag{9}$$

In conventional slender-body theory the term ϕ_x can be ignored in comparison with unity but here it must be retained. Each of the potentials must, of course, also satisfy the condition that its radial derivative vanishes at the tube wall, i.e.,

$$\phi_r = \psi_r = \lambda_r = \beta_r = \gamma_r = 0 \quad \text{at } r = R \tag{10}$$

The forces and moments to be calculated later will be obtained by integrating pressures over the body. Hence, consider the unsteady form of Bernoulli's equation:

$$C_p = -2\Phi_x - 2(\Phi_t/U) - \Phi_r^2 - (1/r^2)\Phi_\theta^2 - \Phi_x^2 \tag{11}$$

The pressure must be determined on the body, and consequently Eq. (11) must be evaluated at r as given by Eq. (2). Upon substituting Eq. (6) and expanding according to the rule given in Eq. (8), the following result is obtained:

$$C_p = -2\phi_x - \phi_x^2 - \phi_r^2 - 2\cos\theta\{\delta[\psi_x(1 + \phi_x) + \psi_r\phi_r - (x - a)\phi_{xr}(1 + \phi_x) - (x - a)\phi_r\phi_{rr}] + \dot{\delta}[\lambda_x(1 + \phi_x) + \psi/U + \phi_r\lambda_r] + z_0[\beta_x(1 + \phi_x) + \phi_r\beta_r + \phi_{xr}(1 + \phi_x) + \phi_r\phi_{rr}] + \dot{z}_0[\gamma_x(1 + \phi_x) + \beta/U + \phi_r\gamma_r]\} \tag{12}$$

Here $\ddot{\delta}$ and $\ddot{z}_0$ terms have been omitted since they lead only to added mass type forces which are determined in Ref. 3. Upon substituting the boundary conditions, Eq. (9), some simplifications result and the partial derivatives of the pressure coefficient with respect to the various attitudes and motion become:

$$\begin{gathered}\partial C_p/\partial\delta = -2 \cos\theta[\psi_x - \phi_r - (x - a)\phi_{xr}](1 + \phi_x) \\ \partial C_p/\partial\dot{\delta} = -2 \cos\theta[\lambda_x(1 + \phi_x) + \psi/U - (x - a)\phi_r/U]; \ \partial C_p/\partial z_0 = -2\cos\theta[\beta_x + \phi_{xr}](1 + \phi_x) \\ \partial C_p/\partial\dot{z}_0 = -2\cos\theta[\gamma_x(1 + \phi_x) + \beta/U + \phi_r/U]\end{gathered} \tag{13}$$

Potential due to Thickness

The potential due to thickness ϕ satisfies the boundary conditions prescribed by Eqs. (9) and (10). In addition, it must satisfy the axially symmetric La Place equation:

$$\phi_{xx} + \phi_{rr} + (1/r)\phi_r = 0 \tag{14}$$

Consider a solution of Eq. (14) which consists of sources along the axis integrated over the length l of the body,[5]

$$\phi_1 = -\frac{1}{4\pi}\int_0^l \frac{F(\xi)d\xi}{[(x - \xi)^2 + r^2]^{1/2}} \tag{15}$$

where $F(\xi)$ is the source strength per unit length. This is the usual representation for a slender body of revolution in free air. This representation is, however, inadequate for a body in a tube. What is required instead is a solution of Eq. (14) which is singular along the axis, as Eq. (15) is, but which is regular elsewhere and, at the same time, automatically satisfies the boundary condition at the tube wall. In order to develop a solution of this type assume that it consists of two parts:

$$\phi = \phi_1 + \phi^* \tag{16}$$

where ϕ_1 is precisely of the form given in Eq. (15). The potential ϕ^* can be determined by expanding ϕ_1 as a Fourier integral in the x variable and assuming that ϕ^* consists of nonsingular sinusoidal solutions of Eq. (14) obtained by separating variables, whose coefficients are determined so as to satisfy the boundary condition, Eq. (10). Then the complete solution for ϕ can be shown to be

$$\phi = \phi_1 + \phi^* = -\frac{1}{4\pi}\int_0^l \frac{F(\xi)d\xi}{[(x - \xi)^2 + r^2]^{1/2}} - \frac{1}{2\pi^2}\int_0^l F(\xi)d\xi \int_0^\infty \frac{K_1(\mu R)I_0(\mu r) \cos\mu(x - \xi)d\mu}{I_1(\mu R)} \tag{17}$$

where the I's and K's are modified Bessel functions of the first and second kind, respectively. This solution automatically satisfies the boundary condition at the wall of the tube and will be used to satisfy the boundary condition at the body. Up to this point in the analysis there has been no approximation made, so that Eq. (17) is actually an alternative formulation to that of Levine.[2] At this point the concept of slenderness will be introduced, and this will allow an explicit expression to be derived for the source strength in terms of body geometry. Slender-body theory is based on the concept that the longitudinal dimension is large compared with the lateral dimension. In ordinary slender-body theory this is interpreted to mean that $(x - \xi)$ is large compared to r in the first term of Eq. (17). This interpretation will be retained here for both terms of Eq. (17). In addition, $(x - \xi)$ will be assumed to be large compared to R.

The expansion of the first term in Eq. (17), is well known from conventional slender-body theory.[5] The result is of the form

$$\phi_1 = [F(x)/2\pi] \ln r + g(x) \tag{18}$$

In particular, the radial velocity is

$$\partial\phi_1/\partial r = F(x)/2\pi r \tag{19}$$

The radial derivative of the second term in Eq. (17) is

$$\frac{\partial\phi^*}{\partial r} = -\frac{1}{2\pi^2}\int_0^l F(\xi)d\xi \int_0^\infty \frac{K_1(\mu R)I_1(\mu r)\cos\mu(x-\xi)\mu d\mu}{I_1(\mu R)} \tag{20}$$

Integrating by parts with respect to ξ results in

$$\frac{\partial\phi^*}{\partial r} = -\frac{F(l)}{2\pi^2}\int_0^\infty \frac{K_1(\mu R)I_1(\mu r)\sin\mu(l-x)d\mu}{I_1(\mu R)} - \frac{F(0)}{2\pi^2}\int_0^\infty \frac{K_1(\mu R)I_1(\mu r)\sin\mu x d\mu}{I_1(\mu R)} - \frac{1}{2\pi^2}\int_0^l F'(\xi)d\xi \int_0^\infty \frac{K_1(\mu R)I_1(\mu r)\sin\mu(x-\xi)d\mu}{I_1(\mu R)} \tag{21}$$

In the first term let $p = \mu(l - x)$. Then, applying the slenderness concept, the Bessel functions can be approximated by the first term in their Taylor expansions. The result for the first term is

$$-\frac{1}{2\pi^2}F(l)\frac{r}{R^2}\int_0^\infty \frac{\sin p dp}{p} = -\frac{F(l)r}{4\pi R^2}$$

Similarly, the second term becomes $-F(0)r/4\pi R^2$. In the third term it is necessary to break the ξ integration up into two parts such that in the first part the limits go from 0 to x, and in the second part from x to l. Then in the first of these integrals substitute $\mu(x - \xi) = p$, while in the second substitute $\mu(\xi - x) = p$. Expansion of the Bessel functions then finally yields for all three terms in Eq. (21):

$$\partial\phi^*/\partial r = -rF(x)/2\pi R^2 \tag{22}$$

The complete radial velocity consists of the sum of Eq. (19) and Eq. (22) and is seen to be

$$\partial\phi/\partial r = [F(x)/2\pi](1/r - r/R^2) \tag{23}$$

The axial velocity may be determined in the same way and is found to be

$$\frac{\partial\phi^*}{\partial x} = \frac{1}{2\pi R^2}\int_0^x F(\xi)d\xi - \frac{1}{2\pi R^2}\int_x^l F(\xi)d\xi \tag{24}$$

It should be observed that $\phi_x{}^*$ is of the order Fl/R^2. This may be contrasted with the order of ϕ_{1x} which, from Eq. (18), or more explicitly from Ref. 5, is seen to be of order $F\ln\epsilon/l$. It follows from the ordering hypothesis that ϕ_{1x} is negligible in comparison with $\phi_x{}^*$, and consequently Eq. (24) actually comprises the total ϕ_x. What has just been stated in mathematical terms can be restated in physical terms as follows: When a body is placed in free air the air accelerates in going around the body creating the axial perturbation velocity ϕ_{1x} which may be said to be due directly to body thickness. When the same body is placed in a tube the acceleration occurs for two reasons: the body thickness as before, and also the narrowing of the annular passage. The statement that ϕ_{1x} is negligible compared with $\phi_x{}^*$ merely says that in the tube the blockage effect is more significant than the direct thickness effect.

Although ϕ_{1x} is negligible compared with $\phi_x{}^*$ over almost all of the body, the slender-body assumption ceases to be valid near the tips of the body, so that in that vicinity it may be necessary to retain ϕ_{1x}. This refinement can be ignored in dealing with over-all lateral forces, but may be of importance for determining pressure distributions.

It should be observed that Eq. (23) and (24) together satisfy the axially symmetric La Place equation, Eq. (14), identically. The significance of this point will be brought out subsequently; but first substitute these equations into the boundary condition on the body, Eq. (9), which yields the following integral equation for the source strength,

$$F(x)[1 - \alpha(x)] = S_0'\left[1 + \frac{1}{2S_R}\int_0^x F(\xi)d\xi - \frac{1}{2S_R}\int_x^l F(\xi)d\xi\right] \tag{25}$$

This equation may be contrasted with the corresponding equation for the source strength in the case of conventional slender-body theory (free air case). Equation (25) reduces to the conventional one by allowing $S_R \to \infty$. In the limit it is seen that the source strength F becomes equal to S_0'. It is seen that in the conventional case the local source strength depends only on a local geometric property of the body. In fact, this strictly local dependence is frequently thought of as an inherent property of slender-body theory. In the tube, on the other hand, local dependence is no longer valid because of the necessary retention of the factor $(1 + \phi_x)$ in the boundary condition. Because of the local dependence in the conventional case it is possible to encompass blunt based bodies within the theory since the presence of the blunt base, which in the strictest sense violates the slender-body assumption, can only affect local flow conditions. On the other hand, in the tube the presence of the base can affect the entire flow field since local dependence no longer applies, and therefore a blunt base can no longer be tolerated within the assumptions of slender-body theory. Equation (25) will therefore be solved only for closed bodies, i.e., for bodies which have the property $S(0) = S(l) = 0$.

Equation (25) can be transformed into a first-order differential equation by substituting $F = G'$. The solution is then readily found to be

$$F(x) = S_0'/(1 - \alpha)^2 \tag{26}$$

Upon substituting into Eqs. (23) and (24) it follows that

$$\begin{aligned}\partial\phi/\partial r &= [S_0'/2\pi(1-\alpha)^2](1/r - r/R^2)\\ \partial\phi/\partial x &= 1/(1-\alpha) - 1\end{aligned} \tag{27}$$

The formula derived here for ϕ_x has a very simple physical interpretation. Consider a one-dimensional channel flow with a body in the channel. Then the equation of continuity states that the product of velocity and cross-sectional area must remain constant at every section, i.e., that

$$(S_R - S_0)U(1 + \phi_x) = S_R U \tag{28}$$

Upon solving for ϕ_x, Eq. (27) follows. It can now be seen from another point of view why blunt-based bodies are inadmissible within the confines of slender-body theory. A blunt base represents an abrupt change in cross-sectional area and the flow through such a passageway clearly cannot be adequately described by a one-dimensional channel flow. The channel flow interpretation together with the fact that Eq. (27) satisfies La Place's equation, suggests a general method for generating the potential of a slender body in a tube for arbitrary body and tube cross sections: Determine ϕ_x from one-dimensional continuity considerations, i.e., from Eq. (28), and then substitute into La Place's equation, yielding the Poisson equation

$$\nabla^2\phi = -(d/dx)\phi_x \tag{29}$$

where ∇^2 denotes the two-dimensional La Placian in the cross-flow variables. Since only axially symmetric bodies in circular tubes will be considered here, the result embodied in Eq. (29) will not be elaborated upon.

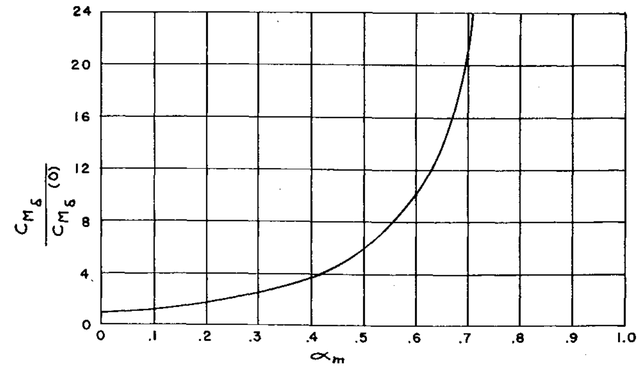

Fig. 2 Augmentation factor for moment coefficient due to angle of attack.

Lateral Potentials

The four quantities $\psi,\lambda,\beta,\gamma$ may be called the lateral potentials although in fact, according to their definitions, Eq. (6), actually $\psi\cos\theta$, $\lambda\cos\theta$, $\beta\cos\theta$, $\gamma\cos\theta$ are truly velocity potentials. According to slender-body theory each of these true potentials must satisfy the two-dimensional La Place equation in the cross-flow variables r and θ. In addition, each of the true potentials must be proportional to $\cos\theta$. Furthermore, each of them must satisfy the boundary condition at the tube wall, Eq. (10). It follows, therefore, that the lateral potentials, $\psi,\lambda,\beta,\gamma$, must all be of the form

$$A(x)(1/r + r/R^2) \tag{30}$$

Upon substituting Eq. (27) into Eq. (9) and evaluating on the body, the boundary conditions on the body for all the lateral potentials become expressed explicitly in terms of the body geometry and the potentials themselves become

$$\begin{aligned}
\psi &= \left\{1 + \frac{(x-a)S_0'}{2S_0}\left(\frac{1+\alpha}{1-\alpha}\right)\right\}\frac{S_0}{\pi}\frac{(1/r + r/R^2)}{(1-\alpha)^2} \\
\lambda &= [S_0(x-a)(1/r + r/R^2)/\pi U(1-\alpha)] \\
\beta &= \left(\frac{-S_0'}{2\pi}\right)\left[\frac{(1+\alpha)(1/r + r/R^2)}{(1-\alpha)^3}\right] \\
\gamma &= -S_0(1/r + r/R^2)/\pi U(1-\alpha)
\end{aligned} \tag{31}$$

Lift and Moment

Upon substituting Eqs. (27) and (31) into Eq. (13), the various pressure coefficients can be evaluated on the body. The lift is obtained by integrating the pressure around the periphery of each section and over the length of the body. Pitching moments about the pivot point are determined by multiplying the pressure by $(x - a)$ and integrating in the same way:

$$\begin{aligned}
\frac{L}{q} &= -\int_0^l dx \int_{-\pi}^{\pi} C_p r_0 \cos\theta d\theta \\
\frac{M}{q} &= -\int_0^l (x-a)dx \int_{-\pi}^{\pi} C_p r_0 \cos\theta d\theta
\end{aligned} \tag{32}$$

According to this definition a nose-down moment is positive.

Each of the x integrals must be integrated by parts at least once. The final formulas are presented below without further derivation

$$\begin{aligned}
\frac{\partial}{\partial\delta}\left(\frac{L}{q}\right) &= \frac{-2}{S_R}\int_0^l \frac{(x-a)(1+\alpha)S_0'^2dx}{(1-\alpha)^5} \\
\frac{\partial}{\partial\delta}\left(\frac{M}{q}\right) &= \frac{-2}{S_R}\int_0^l \frac{(x-a)^2(1+\alpha)S_0'^2dx}{(1-\alpha)^5} - \frac{2}{3}S_R\int_0^l \frac{(3-3\alpha+2\alpha^2)\alpha dx}{(1-\alpha)^3} \\
\frac{\partial}{\partial\dot{\delta}/U}\left(\frac{L}{q}\right) &= 2S_R\int_0^l \frac{(1+\alpha)\alpha dx}{(1-\alpha)^2} \\
(\partial/\partial\dot{\delta}/U)(M/q) &= 0 \\
\frac{\partial}{\partial z_0}\left(\frac{L}{q}\right) &= \frac{2}{S_R}\int_0^l \frac{(1+\alpha)S_0'^2dx}{(1-\alpha)^5} \\
(\partial/\partial z_0)(M/q) &= -(\partial/\partial\delta)(L/q),\ (\partial/\partial\dot{z}_0/U)(L/q) = 0 \\
(\partial/\partial\dot{z}_0/U)(M/q) &= (\partial/\partial\dot{\delta}/U)(L/q)
\end{aligned} \tag{33}$$

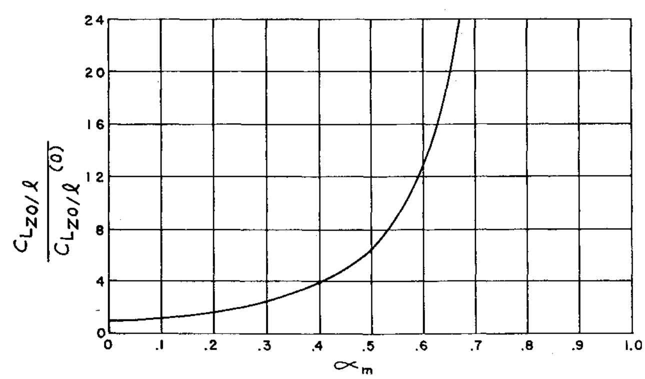

Fig. 3 Augmentation factor for lift due to displacement.

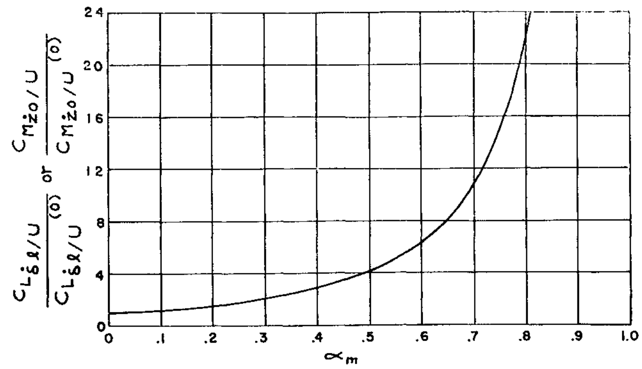

Fig. 4 Augmentation factor for lift coefficient due to rate of change of angle of attack or moment coefficient due to heave rate.

Stability Derivatives of the Ellipsoid of Revolution

As an illustrative example the stability derivatives of an ellipsoid will be worked out in detail. The equation describing the cross-sectional area distribution of an ellipsoid of revolution whose axis lies in the interval $-1 \leq x \leq 1$ is

$$S_0 = S_m(1 - x^2) \tag{34}$$

Dividing both sides by S_R leads to

$$\alpha = \alpha_m(1 - x^2) \tag{35}$$

The pivot point, a, will be taken to lie at the center of the body, i.e., at $x = 0$. Then, since the body is symmetric fore and aft,

$$C_{L_\delta} = 0,\ C_{M_{\dot{\delta}l/U}} = 0,\ C_{M_{z_0/l}} = 0,\ C_{L_{\dot{z}_0/U}} = 0 \tag{36}$$

On the other hand,

$$C_{M_\delta} = -\frac{1}{24\alpha_m(1-\alpha_m)^2}\left\{-33 + 70\alpha_m - 16\alpha_m^3 + 3(11 - 20\alpha_m + 16\alpha_m^2)\frac{\sin^{-1}(\alpha_m)^{1/2}}{[\alpha_m(1-\alpha_m)]^{1/2}}\right\}$$

$$C_{L_{\dot{\delta}l/U}} = \frac{2}{\alpha_m(1-\alpha_m)}\left\{2-\alpha_m-(2-3\alpha_m)\frac{\sin^{-1}(\alpha_m)^{1/2}}{[\alpha_m(1-\alpha_m)]^{1/2}}\right\}$$

$$C_{L_{z_0/l}} = \frac{1}{12(1-\alpha_m)^3}\left\{-3+50\alpha_m-48\alpha_m^2+16\alpha_m^3+3(1+4\alpha_m)\frac{\sin^{-1}(\alpha_m)^{1/2}}{[\alpha_m(1-\alpha_m)]^{1/2}}\right\}$$

$$C_{M_{\dot{z}_0/U}} = C_{L_{\dot{\delta}l/U}} \qquad (37)$$

By taking the limit $\alpha_m \to 0$, each of these formulas reduces to

$$C_{M_\delta} \to -\tfrac{4}{3};\ C_{L_{\dot{\delta}l/U}} = C_{M_{\dot{z}_0/U}} \to \tfrac{4}{3};\ C_{L_{z_0/l}} \to \tfrac{32}{3}\alpha_m \qquad (38)$$

These limiting values for C_{M_δ}, $C_{L_{\dot{\delta}l/U}}$, and $C_{M_{\dot{z}_0/U}}$ are the correct slender-body values for the ellipsoid in free air as expected. Each of the formulas presented in Eq. (37) has been normalized by the corresponding value presented in Eq. (38), and these normalized formulas are shown plotted in Figs. 2–4. The purpose of presenting the formulas in normalized form is that the plots shown can be considered to be augmentation factors to be applied to free-air values of the stability derivatives to account for the presence of the tube. Thus, experimental values for the free-air stability derivatives may already exist which are more accurate than the slender-body values given in Eq. (38). The free-air values are likely to depend critically on body shape, but the tube wall augmentation factors are likely to depend more on the blockage area than on the shape of the body. Hence the factors presented in Figs. 2–4, while ostensibly valid only for an ellipsoid, are probably fairly accurate for almost any symmetrical body pivoted about its center.‡ Of course, it is not feasible to determine an experimental free-air value for the derivative $C_{L_{z_0/l}}$ because this derivative is identically zero in free air, and so for this case it may be necessary to rely entirely on the value given in Eq. (38).

‡ Subsequent calculations for a cylinder with ogival ends fore and aft show only small deviations from these curves.

Results and Conclusions

Formulas for the static and dynamic aerodynamic stability derivatives of a slender body traveling in a tube have been obtained and applied to an ellipsoid as an illustrative example. In general the results of these calculations show that all the stability derivatives which are present for the same body in free air are also present for the body in the tube and their magnitudes are augmented. Furthermore, a force is present which does not exist in free air at all; this is the force of attraction to the wall due to proximity to the wall. This force of attraction increases as the body moves closer to the wall, a statically unstable situation.

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